# Ternary generalizations of Grassmann algebra

**Viktor Abramov**

Puhta Matemaatika Instituut, Tartu Ülikool (Institute of Pure Mathematics, University of Tartu), Vanemuise 46, Tartu, Eesti (Estonia).

**Abstract.** We propose the ternary generalization of the classical anti-commutativity and study the algebras whose generators are ternary anti-commutative. The integral over an algebra with an arbitrary number of generators $N$ is defined and the formula of a change of variables is proved. In analogy with the fermion integral we define an analogue of the Pfaffian for a cubic matrix by means of Gaussian type integral and calculate its explicit form in the case of $N = 3$.

**Keywords:** Ternary anti-commutativity, Grassmann algebra, Pfaffian.

## 1 Introduction

If $\mathcal{A}$ is an algebra with composition law $(a, b) \to a \cdot b$ then its composition law is said to be anti-commutative if $a^2 = 0, \forall a \in \mathcal{A}$. The most known examples of an algebra with anti-commutative multiplication are provided by Lie algebras. The first natural generalization of anti-commutative multiplication is to increase the number of arguments, i.e. to consider the algebras whose composition law involves $n$ elements keeping the order of nilpotency the same. This generalization was studied by Mal'tsev and his collaborators in 1960's.

Another possible generalization is to increase the order of nilpotency and this generalization is our main concern in this paper. It is obvious that this generalization requires algebras with at least ternary composition law. Thus if $\mathcal{T}$ is an algebra with ternary multilplication $(a, b, c) \to a \cdot b \cdot c \in \mathcal{T}$ then we shall call its multiplication *ternary anti-commutative* if $a^3 = 0,\ \forall a \in \mathcal{T}$. Then from the identities $(a + b)^3 = 0$ and $(a + b + c)^3 = 0$, where $a, b, c$ are an arbitrary elements of the algebra $\mathcal{T}$, it follows immediately that

$$a \cdot b \cdot c + b \cdot c \cdot a + c \cdot a \cdot b + c \cdot b \cdot a + a \cdot c \cdot b + b \cdot a \cdot c = 0.$$

The left-hand side of the above identity suggests to introduce in analogy with the classical anti-commutativity the ternary anti-commutator

$$\{a_1, a_2, a_3\} = \sum_{\sigma \in S_3} a_{\sigma(1)} \cdot a_{\sigma(2)} \cdot a_{\sigma(3)}. \tag{1}$$

If $a, b, c$ are the elements of some ternary algebra then we shall call them ternary anti-commutative elements if $\{a, b, c\} = 0$.

In this paper we study the algebras whose generators are ternary anti-commutative. These algebras may be viewed as an analogues of Grassmann algebra. Therefore we use the term ternary Grassmann algebra for them. Since classical Grassmann algebras have played an essential role in supersymmetric field theories there have been made attempts to find an applications of ternary Grassmann algebras in field theories. The ternary Grassmann algebra with ternary defining relations is used in [3] to construct the operators which are more fundamental then the operators of supersymmetry. The algebra with one ternary anti-commutative generator is used in [6] to construct the $Z_3$-graded quantum space and in [1], [7] to generalize the algebras of supersymmetries. Therefore we hope that other ternary structures such as ternary generalizations of Clifford and Lie algebras will be a ground for the field theories with new kind of symmetries.

# 2 Ternary Grassmann algebras

We begin this section with the general definition of the ternary Grassmann algebra. An associative algebra over the field $\mathbf{C}$ generated by $\theta_1, \theta_2, \ldots, \theta_N$ is called *ternary Grassmann algebra* (TGA) if its generators satisfy the following condition of ternary anti-commutativity:

$$\{\theta_A, \theta_B, \theta_C\} = 0, \qquad \forall A, B, C = 1, \ldots, N. \tag{2}$$

Since each classical Grassmann algebra is TGA we define *proper ternary Grassmann algebra* (PTGA) as a ternary Grassmann algebra whose generators satisfy the additional condition $\theta_A^2 \neq 0$, $\forall A = 1, \ldots, N$.

The generators of any TGA are cubic nilpotent, i.e. $\theta_A^3 = 0$, $\forall A = 1, \ldots, N$. This property follows from (2) when $A = B = C$. PTGA can be endowed with the $Z_3$-grading defined as follows: each generator $\theta_A$ has grade 1 and the grade of any monomial equals its degree with respect generators $\theta_A$ modulo 3.

We get the simplest example of a PTGA when $N = 1$. This algebra is a three dimensional vector space over the field $\mathbf{C}$ and it is spanned by the monomials $1, \theta, \theta^2$, where $\theta$ is the generator. This algebra was used in [6] to construct the $Z_3$-graded quantum space.

In order to have an explicit construction of PTGA when $N > 1$ one ought to find the defining commutation relations which are consistent with (2). In this paper we shall describe two ways of solving the condition (2) of ternary anti-commutativity.

### 2.1 Ternary Grassmann algebra with binary relations

Let us assume some binary commutation relations between the generators $\theta_1, \ldots, \theta_N$ of PTGA. Let these binary relations be of the form

$$\theta_A \theta_B = q_{AB}\, \theta_B \theta_A,$$

where $q_{AB}$ are complex numbers such that $q_{AB} \neq 0$ for each pair of indices $(A, B)$. It is clear that $q_{AB} = 1$ for $A = B$ since $\theta_A^2 \neq 0$ and $q_{AB} = q_{BA}^{-1}$. Putting these binary commutation relations into the condition (2) one obtains

$$1 + q_{BA} + q_{CB} + q_{CA}\, q_{BA} + q_{CA}\, q_{CB} + q_{CB}\, q_{CA}\, q_{BA} = 0. \qquad (3)$$

If $B = C$, $B \neq A$ then the above condition takes on the form

$$1 + q_{BA} + q_{BA}^2 = 0,$$

which clearly shows that $q_{AB}$ is the cubic root of unit. Here, we have a choice between $j$ and $j^2$, where $j = e^{\frac{2\pi i}{3}}$. Let us choose $q_{AB} = j$ for $A > B$ and $q_{AB} = j^2$ for $A < B$. It is obvious that another choice leads just to the same structure. Now we are able to define the *PTGA with binary relations* between its generators. This algebra is an associative algebra over the field $\mathbf{C}$ generated by $\theta_1, \ldots, \theta_N$ which are subjected to the following commutation relations:

$$\theta_A \theta_B = q_{AB}\, \theta_B \theta_A, \qquad \theta_A^3 = 0, \qquad (4)$$

where

$$q_{AB} = \begin{cases} 1, & A = B \\ j, & A > B \\ j^2, & A < B \end{cases} \qquad (5)$$

Let us denote this PTGA with binary relations by $\mathcal{G}_B^N$. In order to make the structure of the algebra $\mathcal{G}_B^N$ more transparent we shall use generators with conjugate indices defined as $\theta_{\bar{A}} = \theta_A^2$. From commutation relations (4) it follows then

$$\theta_A \theta_{\bar{A}} = \theta_{\bar{A}} \theta_A = 0,\ \theta_A \theta_{\bar{B}} = \bar{q}_{AB}\, \theta_{\bar{B}} \theta_A,\ \theta_{\bar{A}} \theta_{\bar{B}} = q_{AB}\, \theta_{\bar{B}} \theta_{\bar{A}},\ \theta_{\bar{A}}^2 = 0. \tag{6}$$

It is helpful to introduce notations which are similar to Kostant ones for the classical Grassmann algebra. Let $\mathcal{N} = \{1, \ldots, N\}$ and $J = (A_1, \ldots, A_k)$ be a subset of $\mathcal{N}$. We associate two monomials $\theta_J$ and $\theta_{\bar{J}}$ to each subset $J \subset \mathcal{N}$ defining them as follows:

$$\theta_J = \theta_{A_1} \theta_{A_2} \ldots \theta_{A_k}, \quad \theta_{\bar{J}} = \theta_{\bar{A}_1} \theta_{\bar{A}_2} \ldots \theta_{\bar{A}_k}. \tag{7}$$

If $J = \emptyset$ then as usual $\theta_\emptyset = 1$. Then the algebra $\mathcal{G}_B^N$ is a vector space over $\mathbf{C}$ spanned by the monomials $\theta_J\, \theta_{\bar{K}}$ such that $J \cap K = \emptyset$. Thus an arbitrary element $f(\theta)$ of $\mathcal{G}_B^N$ can be expressed as

$$f(\theta) = \sum_{J \cap K = \emptyset} \alpha_{J\bar{K}}\, \theta_J\, \theta_{\bar{K}}, \tag{8}$$

where $\alpha_{J\bar{K}}$ are complex numbers.

The number of subsets $J \subset \mathcal{N}$ consisting of $k$ elements is $C_N^k$. Since the subset $K \subset \mathcal{N}$ matches $J$ if $J \cap K = \emptyset$ it is obvious that $K \subset \mathcal{N} \backslash J$ and the number of such subsets is $2^{N-k}$. Thus the total dimension of $\mathcal{G}_B^N$ is $\sum_{k=1}^N C_N^k\, 2^{N-k} = 3^N$. The highest degree monomial of the algebra $\mathcal{G}_B^N$ is the monomial $\theta_{\bar{1}} \theta_{\bar{2}} \ldots \theta_{\bar{N}} = \theta_1^2 \theta_2^2 \ldots \theta_N^2$.

If we write an arbitrary element $f(\theta)$ in the form $f(\theta) = \alpha + O(\theta)$, where $O(\theta)$ stands for the terms each containing at least one generator $\theta_A$ then it can be shown that there exists the inverse element $f^{-1}(\theta)$ if and only if $\alpha \neq 0$.

### 2.2. Ternary Grassmann algebra with ternary relations

Another way to solve the conditions of ternary anti-commutativity (2) is to assume some ternary commutation relations between generators $\theta_A$. Since the cyclic subgroup $Z_3$ of the group $S_3$ has the representation by cubic roots of unit it seems natural in analogy with the ordinary anti-commutativity to construct ternary commutation relations by means of the action of the

cyclic group $Z_3$ on the indices of the corresponding variables. This idea was first proposed by R. Kerner and in this subsection we briefly describe the structure of the corresponding ternary Grassmann algebra. More detailed description of the ternary Grassmann algebra with ternary relations and its applications can be found in [2], [3].

The *PTGA with ternary commutation relations* is an associative algebra over the field $\mathbf{C}$ generated by $\theta_1, \theta_2, \ldots, \theta_N$ which are subjected to the following ternary defining relations:

$$\theta_A\theta_B\theta_C = j\,\theta_B\theta_C\theta_A. \tag{9}$$

Let us denote the PTGA with ternary relations by $\mathcal{G}_T^N$. The above ternary defining relations (9) are based on the idea of the action of the cyclic group $Z_3$ in the sense that each cyclic permutation of the indices in the product $\theta_A\,\theta_B\,\theta_C$ is accompanied by the multiplication by cubic root of unit accordingly to the representation of $Z_3$. It is obvious that the generators of $\mathcal{G}_T^N$ satisfy the conditions (2) of ternary anti-commutativity.

It should be noted here that there are no any relations between the binary products $\theta_A\theta_B$ of generators of $\mathcal{G}_T^N$ that is they are linearly independent entities. The immediate corollary from the above definition is that any product of four or more generators must vanish. Here is the proof:

$$\begin{aligned}(\theta_A\theta_B\theta_C)\theta_D = j\theta_B(\theta_C\theta_A\theta_D) &= j^2(\theta_B\theta_A\theta_D)\theta_C\\ &= \theta_A(\theta_D\theta_B\theta_C) = j\theta_A\theta_B\theta_C\theta_D.\end{aligned}$$

Now, as $(1-j) \neq 0$, one must have $\theta_A\theta_B\theta_C\theta_D = 0$. Thus the monomials $\theta_A\,\theta_B^2$ are the highest degree monomials of the algebra $\mathcal{G}_T^N$. The dimension of the PTGA with ternary relations is $N(N+1)(N+2)/3+1$, because we have $N$ generators, $N^2$ independent products of two generators, $N(N-1)$ independent ternary expressions with two generators equal and one different, and $N(N-1)(N-2)/3$ ternary products with all the three generators different; finally, the numbers give an extra dimension. Any cube of a generator is equal to zero; the odd permutation of factors in a product of three leads to an independent quantity.

# 3 Integration

The aim of this section is to define the derivatives and integral over the PTGA generated by an arbitrary number of generators $N$. We shall also

establish and prove the formula of a change of variables in the integral over the ternary Grassmann algebra. Though the definitions of derivatives and integrals are just the same in both cases of $\mathcal{G}_B^N$ and $\mathcal{G}_T^N$ we shall always assume in this section that we are considering the algebra $\mathcal{G}_B^N$.

### 3.1 Derivatives and integral

Using the notations of the subsection 2.1 we define the derivatives with respect generators $\theta_A$ by the following set of rules:

$$\partial_A(\theta_B) = \delta_{AB}, \quad \partial_A(\theta_{\bar{B}}) = (1 + j^2)\,\delta_{AB}\theta_B. \tag{10}$$

It is also helpful to define the derivatives with respect squares of generators $\theta_{\bar{A}} = \theta_A^2$ as follows:

$$\partial_{\bar{A}}(\theta_B) = 0, \quad \partial_{\bar{A}}(\theta_{\bar{B}}) = \delta_{AB}. \tag{11}$$

It is easy to establish the relation between these derivatives and the second order derivatives

$$\partial_{\bar{A}} = (1 + j)\,\partial_A^2.$$

Clearly that each derivative $\partial_A$ is an operator of grade 2 and each derivative $\partial_{\bar{A}}$ is an operator of grade 1. Straightforward computation shows that derivatives satisfy the following commutation relations:

$$\partial_A^3 = 0, \quad \partial_A\,\partial_{\bar{A}} = \partial_{\bar{A}}\,\partial_A = 0, \qquad \partial_A\,\partial_B = q_{AB}\,\partial_B\,\partial_A,$$

$$\partial_A\,\partial_{\bar{B}} = \bar{q}_{AB}\,\partial_{\bar{B}}\,\partial_A, \quad \partial_{\bar{A}}\,\partial_{\bar{B}} = q_{AB}\,\partial_{\bar{B}}\,\partial_{\bar{A}}, \qquad \partial_{\bar{A}}^2 = 0.$$

From the above formulae it follows that derivatives $\partial_A$, $\partial_{\bar{A}}$ are ternary anticommutative, i.e.

$$\{\partial_A, \partial_B, \partial_C\} = 0, \quad \{\partial_{\bar{A}}, \partial_{\bar{B}}, \partial_{\bar{C}}\} = 0.$$

Integral over the PTGA generated by one generator was defined and studied in [6]. We extend there given definition to the PTGA with $N$ generators and prove the formula of a change of variables. The integral of an arbitrary element $f(\theta) \in \mathcal{G}_B^N$ with respect $\theta_A$ is defined by the formula

$$\int d\theta_A\; f(\theta) = \partial_{\bar{A}}(f(\theta)). \tag{12}$$

As usual the multiple integral is to be understood as the repeated integral.

Note that integration with respect all generators in the case of the PTGA with ternary relations $\mathcal{G}_T^N$ is always trivial since the highest degree monomials have the form $\theta_A\theta_B^2$. Integration with respect all generators in the case of PTGA with binary relations $\mathcal{G}_B^N$ yields the coefficient at the highest degree monomial. Thus

$$\int \mathcal{D}\theta\ f(\theta) = \alpha_{\bar{1}\bar{2}\ldots\bar{N}}, \tag{13}$$

where $\mathcal{D}\theta = d\theta_1\, d\theta_2 \ldots d\theta_N$ and $\alpha_{\bar{1}\bar{2}\ldots\bar{N}}$ is the coefficient at the monomial $\theta_{\bar{1}}\theta_{\bar{2}}\ldots\theta_{\bar{N}}$.

Let $\vartheta_1, \vartheta_2, \ldots, \vartheta_N$ be another system of generators of the algebra $\mathcal{G}_B^N$ and generators $\theta_1, \theta_2, \ldots, \theta_N$ are expressed in terms of $\vartheta_1, \vartheta_2, \ldots, \vartheta_N$ as follows:

$$\theta_A = \sum_{B=1}^{N} \alpha_{AB}\, \vartheta_B + O_A(\theta), \tag{14}$$

where $O_A(\theta)$ means terms containing more than one generator and the determinant of the matrix $A = (\alpha_{AB})$ differs from zero. If $T(\theta, \vartheta)$ is the Jacobian matrix of the above transformation then we define the *Jacobian* $J(\theta, \vartheta)$ by the formula

$$J(\theta, \vartheta) = det^{-2}(T(\theta, \vartheta)). \tag{15}$$

It can be proved that

$$\int \mathcal{D}\theta\ f(\theta) = \int \mathcal{D}\vartheta\ J(\theta, \vartheta)\tilde{f}(\vartheta). \tag{16}$$

It should be noted that in contrast to fermion integral where determinant of the Jacobian matrix appears in the formula of a change of variables in power $-1$ in the above formula it has the power $-2$.

Now we turn to the proof of the formula (16). It is based on the observation that if two systems of generators of the algebra $\mathcal{G}_B^N$ are related by the formulae (14) then this imposes (in the contrast to the classical Grassmann algebra) very strong restrictions on the coefficients of the expressions at the right-hand sides of (14). Since these restrictions lead to the bulky conditions on the coefficients we produce only the conditions for the entries of the matrix $A$ and prove the formula (16) when the right-hand side expressions of (14) contain only linear terms with respect generators. The entries of $A$ must satisfy the following conditions:

$$\begin{cases} \alpha_{AD}\alpha_{BC} = 0 & A < B,\, C < D \\ \alpha_{AD}\alpha_{BD} = 0 & A < B, \text{ (no summation!)}. \end{cases} \tag{17}$$

Taking into account also the condition $det\, A \neq 0$ guaranteeing the linear independce of the new generators we conclude that the matrix $A$ is a diagonal matrix. Thus we have

$$\int \mathcal{D}\vartheta\, J(\theta,\vartheta)\, \tilde{f}(\vartheta) = \int \mathcal{D}\vartheta\, \prod_{A=1}^{N} (\alpha_{AA}^{-2})\, \tilde{f}(\vartheta) = \int \mathcal{D}\theta\, f(\theta),$$

and this ends the proof.

### 3.2 Pfaffian of a cubic matrix

It is well-known ([5]) that the fermion integral of Gaussian type over the even dimensional classical Grassmann algebra can be used to derive the Pfaffian of a skew-symmetric square matrix. Replacing the notion of a skew-symmetric square matrix by its cubic analogue and making use of the integral over PTGA with binary relations we define the Pfaffian of a cubic matrix and calculate its explicit form in the dimension $N = 3$.

Let $\Omega(\theta)$ be a cubic form

$$\Omega(\theta) = \frac{1}{3}\, \omega_{ABC}\, \theta_A \theta_B \theta_C, \tag{18}$$

with the coefficients satisfying the relations

$$\omega_{ABC} = \bar{q}_{AB}\, \omega_{BAC},\ \ \omega_{ABC} = \bar{q}_{BC}\, \omega_{ACB},$$

if there are at least two different indices in the triple $(A, B, C)$ and

$$\omega_{AAA} = 0.$$

The coefficients $\omega_{ABC}$ of the cubic form $\Omega(\theta)$ can be considered as the entries of a cubic $N \times N \times N$-matrix we shall denote by $\Omega$. From the above relations it follows that the entries of the cubic matrix $\Omega$ satisfy the relations

$$\omega_{ABC} + \omega_{BCA} + \omega_{CAB} + \omega_{BAC} + \omega_{ACB} + \omega_{CBA} = 0, \tag{19}$$

for any triple of indices $A, B, C$. The property (19) may be considered as a cubic generalization of the notion of skew-symmetric square matrix. We define the *Pfaffian* of this cubic matrix by the following integral:

$$Pf_{cub}(\Omega) = \int \mathcal{D}\theta\ e^{\Omega(\theta)}. \tag{20}$$

It is not a surprise that the above integral leads to a non-trivial result only when the number $N$ of generators is a number divisible by 3. Thus the dimension $N = 3$ is the lowermost dimension providing a non-trivial result. Let us find the Pfaffian of a cubic matrix in this case. The cubic form (18) then takes on the form

$$\begin{aligned}\Omega(\theta) \;=\; & 2\,\omega_{123}\,\theta_1\theta_2\theta_3 + \omega_{112}\,\theta_1^2\theta_2 + \omega_{122}\,\theta_1\theta_2^2 + \omega_{113}\,\theta_1^2\theta_3 \\ & + \omega_{133}\,\theta_1\theta_3^2 + \omega_{223}\,\theta_2^2\theta_3 + \omega_{233}\,\theta_2\theta_3^2.\end{aligned}$$

Making use of the definition of the integral over ternary Grassmann algebra one obtaines the following homogeneous polynomial for the Pfaffian:

$$\begin{aligned}Pf_{cub}(\Omega) \;=\; & \int \mathcal{D}\theta\; e^{\Omega(\theta)} = \int \mathcal{D}\theta\; (1 + \Omega(\theta) + \frac{1}{2!}\Omega^2(\theta)) \\ & = 4\,\omega_{123}^2 - \omega_{211}\omega_{233} - \omega_{221}\omega_{133} - \omega_{311}\omega_{223}.\end{aligned}$$

We end this section by the following speculation. If we would develop the calculus of the cubic matrices based on the ternary Grassmann algebra approach we would define the determinant of the cubic $3 \times 3 \times 3$-matrix $\Omega$ as the third power of the above polynomial $Pf_{cub}(\Omega)$ and then the determinant would be a sum of products each containing six entries of the cubic matrix $\Omega$. This suggests that the group $S_3$ of permutations of three elements is likely to play an essential role in the definition of the detrminant of cubic matrices.

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